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RELIABILITY BOUNDS AND CRITICAL TIME
FOR THE BERNSTEIN DISTRIBUTION

by

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ABSTRACT

The Bernstein distribution has been successfully used to model the probabilistic characteristics of non-stationary linear random wear or deterioration processes. In this paper, the approach of obtaining $s$–confidence bounds for the reliability is proposed. Then the point estimate for the critical time is given, and $s$–confidence intervals for the critical time of Bernstein distribution is constructed. Finally numerical examples are used to illustrate the procedure.
1. INTRODUCTION

The Bernstein distribution (inverted normal or lognormal) has been proposed as a lifetime model to describe the non-stationary linear wear processes\(^1\), and its statistical characteristics and properties has been studied by Ahmad & Sheikh\(^2\) and Sheikh, Ahmad & Younas\(^3\).

In this paper, two issues concerning the confidence intervals related to Bernstein distribution are addressed. The first one is the construction of reliability bounds, which is useful to designer who may need the lower reliability bound. The second one is obtaining confidence intervals for the critical time of hazard function, which is useful in conducting the burn-in process. These confidence intervals are derived from the confidence intervals of the parameters. Illustrative examples are used to demonstrate the computation procedure.

\textit{Notation}

c, \alpha \quad \text{the parameters of Bernstein distribution}
\hat{(\cdot)} \quad \text{the estimate of } (\cdot).
\text{MLE} \quad \text{maximum likelihood estimate}
\Phi(\cdot) \quad \text{cdf of standard normal distribution}
g'(\cdot) \quad \text{the first derivative of function } g(\cdot)
\log(.) \quad \text{natural log base } e \text{ of } (\cdot)
t^*(c, \alpha) \quad \text{critical time of hazard function}
(\cdot), (\cdot) \quad \text{lower and upper confidence intervals of } (\cdot)
\gamma_1, \gamma_2 \quad \text{arbitrary number, } 0 \leq \gamma_i \leq 1, i=1, 2
2. PARAMETERS INFERENCES

Suppose that the stochastic phenomena of damage propagation is considered as:

\[ P(t) = P_0 \cdot t^\beta, \quad t \in T \]  \hspace{1cm} (1)

where \( P_0 = P(1) \) is a deterministic initial value and \( \beta \) is normally distributed with mean \( \mu_\beta \) and variance \( \sigma^2_\beta \).

Eq.(1) can be rewritten as

\[ \log P(t) = \log P_0 + \beta \cdot \log t, \quad t \in T \]  \hspace{1cm} (2)

For the failure critical level \( P(t) \geq P_1 \), we have

\[ \log T = \frac{\log P_1 - \log P_0}{\beta}. \]

The random variable \( \log T \) is a Bernstein distribution with parameters \( c = (\log P_1 - \log P_0)/\mu_\alpha \) and \( \alpha = \sigma^2_\beta/\mu^2_\beta \). The pdf and reliability function of \( T \) are

\[ f(t) = \frac{c}{\sqrt{2\pi\alpha \cdot t \cdot (\log t)^2}} \cdot \exp\{ -\frac{1}{2\alpha} \cdot \left[ 1 - \frac{c}{\log t} \right]^2 \} \]  \hspace{1cm} (3)

\[ R(t) = 1 - \Phi\left[ \frac{1}{\sqrt{\alpha}} \cdot \left( 1 - \frac{c}{\log t} \right) \right] \]  \hspace{1cm} (4)

The maximum likelihood estimates of \( (c, \alpha) \) is given as\(^2\).
\[ \hat{c} = 1 / \left[ \frac{\sum_{i} \frac{1}{\log t}}{n} \right] \] 

(5a)

\[ \hat{\alpha} = \frac{1}{n} \sum_{i} \left( \frac{1}{\log t} - \frac{1}{c} \right)^2 \] 

(5b)

for a given data set \( \{ t_1, t_2, \ldots, t_n \} \).

The asymptotic variances and covariance of \( \hat{c} \) and \( \hat{\alpha} \) are,

\[
\text{var}(\hat{c}) = \frac{\alpha c}{n} \\
\text{var}(\hat{\alpha}) = 2\alpha^2(1+2\alpha)/n \\
\text{cov}(\hat{c}, \hat{\alpha}) = 2\alpha^2 c/n.
\]

(6)

The estimates of variances and covariances are obtained by replacing \((c, \alpha)\) by \((\hat{c}, \hat{\alpha})\).

3. DERIVATION OF CONFIDENCE BOUNDS

For Bernstein distribution, it is clear that the reliability function is not monotone with respect to \( \alpha \) in eq.(4) as shown in figure 1. Therefore, there is no equivariant confidence set for \( R(t) \) \(^4\). However, we can still obtain a confidence bounds for \( R(t) \) as follows.

Suppose that the \( 100 \cdot (1 - \gamma_1) \)% confidence interval of \( \alpha \) and \( 100 \cdot (1 - \gamma_2) \)% confidence interval of \( c \) are given by

\[ [\alpha, \overline{\alpha}] \quad \text{and} \quad [c, \overline{c}] \]

respectively. We have the following results.
Theorem 1  The $100 \cdot (1 - \gamma_1 - \gamma_2)$% level confidence bounds for $R(t)$ of Bernstein distribution is given by the interval

$$\left[ \inf_{\alpha \leq \alpha \leq \bar{\alpha}} R(t; c, \alpha), \sup_{\alpha \leq \alpha \leq \bar{\alpha}} R(t; \bar{c}, \alpha) \right], \forall t \geq 1. \quad (7)$$

Proof: See Appendix.

From figure 1, we observe that $R(t; c, \alpha)$ is monotone decreasing in $\alpha$ for $t \leq c$ and monotone increasing in $\alpha$ for $t \geq c$. Therefore, in practice, we have the following results:

1. for $t \leq c$, $\inf_{\alpha \leq \alpha \leq \bar{\alpha}} R(t; c, \alpha) = R(t; c, \bar{\alpha})$ and $\sup_{\alpha \leq \alpha \leq \bar{\alpha}} R(t; \bar{c}, \alpha) = R(t; \bar{c}, \alpha)$;

2. for $t \geq c$, $\inf_{\alpha \leq \alpha \leq \bar{\alpha}} R(t; c, \alpha) = R(t; c, \alpha)$ and $\sup_{\alpha \leq \alpha \leq \bar{\alpha}} R(t; \bar{c}, \alpha) = R(t; \bar{c}, \alpha)$.

4. CONFIDENCE INTERVALS OF THE CRITICAL TIME

The hazard function for random variable $T$ is given as:

$$h(t) = h(t; c, \alpha) = \frac{f(t)}{R(t)}, \text{ for } t \geq 1. \quad (8a)$$

$$\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \frac{-f_r(t)}{f(t)} = 0 \quad (8b)$$

Figure 2 show Bernstein hazard function for selected values of $\alpha$ and given $c=1$. For all Bernstein distribution $h(t)$ is zero at time 1, increases to a maximum and then decreases, approaching zero at time infinite. From eq.(8a) we have
\[ f'(t; c, \alpha) = -m(t; c, \alpha) \cdot f(t; c, \alpha) \]

where

\[ m(t; c, \alpha) = \frac{1}{t} \cdot \left[ 1 + \frac{2}{\log t} + \frac{c}{\alpha(\log t)^2} \cdot \left(1 - \frac{c}{\log t}\right)\right]. \quad (9) \]

It follows that

\[ h'(t; c, \alpha) = h(t; c, \alpha) \cdot \left[-m(t; c, \alpha) + h(t; c, \alpha)\right], \quad (10) \]

then the solution to eq.(10)

\[ h(t; c, \alpha) - m(t; c, \alpha) = 0 \quad (11) \]

is the critical time \( t^*(c, \alpha) \) of the Bernstein hazard function.

Given \( c=1 \), we solve the critical time by eq.(11) for various values of \( \alpha \) ranging from 0.02 to 2.0. The result is depicted in figure 3. It can be seen that the critical time is a monotonic decreasing function of \( \alpha \). Similarly, given \( \alpha=1 \), we observe that the critical time is a monotonic increasing function of \( c \) as shown in figure 4.

The same approach for derived reliability bounds could be used to obtain the 100 \( \cdot (1 - \gamma_1 - \gamma_2) \)% confidence interval for the critical time \( t^* \) of Bernstein hazard function, that is,

\[ [t^*, t^*] = [t^*(c, \bar{\alpha}), t^*(\bar{c}, \alpha)]. \quad (12) \]

5. ILLUSTRATIVE EXAMPLES

Fatigue data (Virkler et al. \(^5\))
Sixty-eight replicate constant amplitude crack propagation tests were conducted on 2024–T3 aluminum alloy. To check whether Bernstein distribution provides a good model fit for this data set. We first give percentage—percentage plot. Assume the data has the Bernstein distribution \( F_0(t) \), let \( F(t) \) denote the true distribution, then if the model is correct, the curve \{ ( F(t), F_0(t) ) \}, \( t \geq 0 \) falls on the diagonal. We use \( \hat{F}_0(t) \) to estimate \( F_0(t) \), where \( (c, \alpha) \) replaced by the MLE \( (\hat{c}, \hat{\alpha}) \). Moreover, the Mean rank is used to estimate \( F(t) \). The result is shown as figure 5 and note that the points adhere to the diagonal.

The MLE of the parameters are computed from eq.(5) as

\[
\hat{c} = 12.4547 \quad \hat{\alpha} = 3.0306 \times 10^{-5}.
\]

It can be seen that \( \hat{\alpha} \) is small suggesting that the hazard function is IFR. This is to be expected since it is unnecessary to consider burn–in for fatigue. The hazard function for this data set is depicted in figure 6.

For \( \gamma_1 = \gamma_2 = 0.05 \), the 95% confidence intervals of \( c \) and \( \alpha \) can be obtained from eq.(6) as [12.4501, 12.4593] and [2.0119 \times 10^{-5}, 4.0493 \times 10^{-5}] respectively.

We could obtain the 90% reliability bounds by iteratively fixed for some \( t \) to solve:

(i) \[ \text{Maximize} \quad R_{UB}(t) = R(t; 12.4593, \alpha) \]
subject to
\[ 2.0119 \times 10^{-5} \leq \alpha \leq 4.0493 \times 10^{-5}; \text{ and} \]

(ii) \[ \text{Minimize} \quad R_{LB}(t) = R(t; 12.4501, \alpha) \]
subject to
\[ 2.0119 \times 10^{-5} \leq \alpha \leq 4.0493 \times 10^{-5} \]
(i) and (ii) give the upper and lower confidence bounds respectively. The results is depicted in figure 7. It can be seen that these bounds are rather "tight". The $\alpha_{UB}^*(t)$ and $\alpha_{LB}^*(t)$ that give these bounds are plotted in figure 8. It can be seen that $\alpha_{LB}^*(t)$ switches from $\alpha$ to $\alpha$ at $t = \bar{c} = 12.4501$ and $\alpha_{UB}^*(t)$ switches from $\alpha$ to $\alpha$ at $t = \bar{c} = 12.4593$. Also, there is a region where $\alpha_{LB}^*(t) = \alpha_{UB}^*(t) = \alpha$.

**Maintenance data**

Consider the following repair times (hours):

20, 30, 50, 50, 50, 60, 60, 70, 70, 70, 80, 80, 100, 100, 100, 100, 110, 130, 150, 150, 150, 200, 200, 220, 250, 270, 300, 300, 330, 330
400, 400, 450, 470, 500, 540, 540, 700, 750, 880, 900, 1030, 2200, 2450.

First we also give the percentage--percentage plot for these data as shown in figure 9. It reveals Bernstein distribution provide a good fit for this data set. The MLE of the parameters are

$$\hat{c} = 5.0307 \quad \hat{\alpha} = 0.0478$$

and the 95% confidence intervals of $c$ and $\alpha$ are $[4.8890, 5.1724]$ and $[0.0274, 0.0682]$ respectively. The hazard function for this data set is depicted in figure 10. The MLE for the critical time is obtained by solving eq.(11) where $c$ and $\alpha$ are replaced by $\hat{c}$ and $\hat{\alpha}$ as $t^* = 72.29$. We could obtain the 90% confidence intervals of the critical time by using eq.(12). The resulting interval is given by

$$[\hat{t}^*, \hat{t}^*] = [50.99, 115.74].$$
6. CONCLUSION

The approach for constructing the reliability bounds and the confidence interval of the critical time for Bernstein distribution from the confidence intervals of its parameters is presented. It is also applicable to censored data as long as the confidence intervals of the parameters can be established. This approach is easy to do no matter what the target is explicit or implicit function. When the confidence bounds cannot be obtained from some equivariant confidence set, this approach provides a another alternative.

REFERENCES

APPENDIX: PROOF OF THEOREM 1

By the definition of confidence bound in Bickel and Doksum\(^6\), we only need to show that

\[ P[ \inf_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} R(t; \underline{c}, \alpha) \leq R(t; c, \alpha) \leq \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} R(t; \bar{c}, \alpha) ] \geq 1 - \gamma_1 - \gamma_2. \]

It is hold that

\[ \inf_{\Omega} R(t; c, \alpha) \leq R(t; c, \alpha) \leq \sup_{\Omega} R(t; c, \alpha) \text{ whenever } \underline{c} \leq c \leq \bar{c}, \; \underline{\alpha} \leq \alpha \leq \bar{\alpha}, \]

where \( \Omega = \{ (c, \alpha): \underline{c} \leq c \leq \bar{c}, \; \underline{\alpha} \leq \alpha \leq \bar{\alpha} \} \), since the latter is a subset of the former. Thus, from the concept of probability, we have

\[ P[ \inf_{\Omega} R(t; c, \alpha) \leq R(t; c, \alpha) \leq \sup_{\Omega} R(t; c, \alpha) ] \]

\[ \geq P[ \underline{c} \leq c \leq \bar{c}, \; \underline{\alpha} \leq \alpha \leq \bar{\alpha} ] \]

\[ = 1 - \gamma_1 - \gamma_2 \quad \text{since } \underline{c} \text{ and } \underline{\alpha} \text{ are depenent}. \]

From eq.(4), it is easily seen that \( R(t) \) is monotone increasing in \( c \). Hence it follows that

\[ \inf_{\Omega} R(t; c, \alpha) = \inf_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} R(t; \underline{c}, \alpha) \quad \text{and} \quad \sup_{\Omega} R(t; c, \alpha) = \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} R(t; \bar{c}, \alpha). \]
Figure 1: $R(t)$ as a function of $\alpha$. 
Figure 2: Hazard function of the Bernstein distribution for various $\alpha$ and $c=1$. 
Figure 3: Critical time as a function of $\alpha$ for given $c=1$. 
Figure 4: Critical time as a function of \( c \) for given \( \alpha=1 \).
Figure 5: Percentage—Percentage plot for fatigue data
Figure 6: Hazard function for fatigue data.
Figure 7: Confidence bounds of $R(t)$ for fatigue data
Figure 8: The trajectories of $\alpha_{LB}^*$ and $\alpha_{UB}^*$
Figure 9: Percentage–Percentage plot for maintenance data
Figure 10: Hazard function for maintenance data.